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STABILIZABILITY OF SECOND ORDER

BILINEAR SYSTEMS

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1. Introduction:

This report concerns the stabilizability of second order bilinear systems,

$$\dot{x} = Ax + u(Dx + b) \tag{1}$$

where $A, D \in \mathbb{R}^{2x2}$, and $x, b \in \mathbb{R}^2$. While a large literature devoted to the structural properties of such systems has developed over the past decade [1,2,3], it is fair to say that little is understood regarding the qualitative behavior of trajectories of (1). Recently, several authors have investigated the stabilizability of systems of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i(D_i x + b_i) \tag{2}$$

in $\mathbb{R}^n$ [4,6,7]. These papers derive sufficient conditions and construct controllers to stabilize systems which meet specific and quite restrictive requirements. In [6] and [7] a constant pure quadratic feedback law is shown to stabilize (2) when A has purely imaginary eigenvalues, m = 1, and b = 0. In [4], it is demonstrated that a constant linear and quadratic feedback law forces trajectories of (2) into an arbitrarily small bounded neighborhood of the origin, assuming the non-intersection of the varieties $x^T P(D_i x + b_i) = 0$ (for some positive definite symmetric matrix, P) over a region of interest, and a controller design is given when $D_i = b_i c^T$.

It is our opinion that a significant understanding of bilinear systems will not be possible until more systematic analysis has been accomplished. Accordingly, in this report we concentrate on a simple problem in some detail. Specifically, we give necessary and sufficient conditions for the existence of a constant linear feedback controller to stabilize (1). Even given the limited scope of this problem, it is safe to say that the statement of necessary and sufficient conditions is possible only because of recent results in the stability of quadratic systems developed by the authors [5]. This paper depends heavily upon that work.

Problem Statement:

Characterize the properties of the triple (A,b,D) such that for some $c \in \mathbb{R}^2$, for $u \triangleq c^T x$, the resulting second order closed loop system

$$\dot{x} = A_c x + c^T x D x \qquad A_c \triangleq A + bc^T \tag{3}$$

is globally asymptotically stable (g.a.s.).

This problem is completely resolved by Theorem 1, stated below. In the sequel we assume that $b \neq 0$, and $D \neq 0$, and we will adopt the notation $|x,y|$ to indicate the determinant of the 2x2 array $[x,y]$.

Theorem 1: The triple (A,b,D) is stabilizable under constant linear state feedback if and only if either

(i) D has complex conjugate eigenvalues and $|Dx,x|$ has the same sign as $|AD^{-1}b,b|$. If $|AD^{-1}b,b| = 0$ then the special conditions given in Proposition 2, below (section 3), hold;

or (ii) D is singular and its non-zero eigenvector is a stable eigenvector of A. If D is singular with a unique real eigenspace, then the special conditions given in Proposition 4, below (section 4), hold.

After briefly reviewing the properties of the scalar analogue of system (1), we will present some preliminary results and notation in Section 2, then discuss condition (i) of Theorem 1 in Section 3; condition (ii) in Section 4; and finally provide a proof of Theorem 1 by way of summary, in Section 5. A construction for a stabilizing linear constant controller is provided in the proof of each case, and reviewed in the summary.

It is instructive to consider equation (1) when $A = a$, $D = d$, b, x, $u \in \mathbb{R}$, which yields the closed loop system (when $u \triangleq cx$)

$$\dot{x} = (a_c + cdx)x \quad \text{where} \quad a_c \triangleq a + bc.$$

Clearly, when $d \neq 0$ this system has an unstable equilibrium at $x = -a_c/cd$, and the origin is never g.a.s. for any choice of $c \neq 0$. Hence when $a > 0$, (a,b,c) is never stabilizable under constant linear feedback. The restrictive conditions of Theorem 1 seem more intuitively reconcilable when compared to this situation in one

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dimension, where an unstable system can never be stabilized by $u \triangleq cx$.

2. Preliminary Definitions and Results:

Evidently, system (3) is an autonomous quadratic differential equation. In order to characterize the stabilizability of the triple (A,b,D) under constant linear state feedback, we must, therefore, know something about the stability of systems of the form

$$\dot{x} = Ax + c^T x Dx \tag{4}$$

where $A, D \in \mathbb{R}^{2x2}$, $c \in \mathbb{R}^2$. In a previous paper [5], we have given necessary and sufficient conditions for (4) to be g.a.s. To facilitate presentation of these conditions and later results it is useful to introduce a few notational conventions, all developed at greater length in that paper.

Defining the skew-symmetric matrix $J \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we have $x_\perp \triangleq Jx$, the orthogonal complement of x, and $y^T x_\perp = y^T Jx = |x,y|$, where the last symbol denotes the determinant of the array [x,y]. If $a \in \mathbb{R}^2$, let $(a) \triangleq \{x \in \mathbb{R}^2 | x = \alpha a, \alpha \in \mathbb{R}\}$. Let A_s denote the symmetric part of a 2x2 matrix, A. If A is singular then $A = ab^T$ for some $a, b \in \mathbb{R}^2$, and if $A_s = P$ then $x^T Px = 0$ iff x is orthogonal to either a or b.

For ease of discussion, we divide the class of 2x2 real matrices into three sets based upon properties of their eigenvectors as follows: A is said to be <u>nodal</u> if it has at least two distinct eigenvectors; <u>x-critical</u> if (x) is its unique, real eigenspace; and <u>focal</u> if it has no real eigenvectors. Since $|Ax,x| = 0$ iff x is an eigenvector of A, that matrix is focal, x-critical, or nodal iff $|Ax,x|$ is sign definite, semi-definite, or indefinite, respectively. Since $|Ax,x| = x^T JAx$, the identical sign conditions on $[JA]_s$ specify whether A is focal, nodal, or x-critical, as well. Notice that e is an eigenvector of A iff $[JA]_s = [e_\perp a^T]_s$ for some $a \in \mathbb{R}^2$. Some further properties of 2x2 real matrices are presented as facts in Appendix B (and referred to as such throughout the sequel). Finally, we will say that two symmetric matrices agree in sign if they are sign definite or semi-definite, and their non-zero eigenvalues have the same sign.

Using this terminology, we may quote the result of [5] as follows.

Theorem 2 [5]: System (4) is g.a.s. if and only if

(i) A has eigenvalues with non-positive real part;

(ii) $[JD]_s$ and $[D^TJA]_s$ agree in sign;

(iii) one of the following two mutually exclusive conditions holds:

(a) D is focal, and $D^{-1}A$ is either focal or x-critical, where $x \in (c_\perp)$ iff $|A| \neq 0$;

(b) D is x-critical and singular, $|A| \neq 0$, and $A^{-1}D = \gamma D$ for some scalar, γ.

The two distinct cases listed under condition (iii) form a natural framework for the presentation of stabilizability conditions. In Section 3 we discuss the properties of the triple (A,b,D) when D is focal, corresponding to condition (iii.a), above. In Section 4 we consider the case where D is singular, corresponding to condition (iii.b), above. It is an immediate consequence of Theorem 2 that we need consider no other cases.

Lemma 1: If D is not focal and not singular, then system (1) cannot be stabilized by constant linear state feedback.

Proof: If D is nodal and non-singular, then $[JD]_s$ is indefinite, and (3) violates condition (ii) of Theorem 2 for any $c \in \mathbb{R}^2$. If D is x-critical and non-singular, then (2) violates condition (iii) of Theorem 2 for any $c \in \mathbb{R}^2$. □

As a further consequence of Theorem 2, we must choose a linear control law, $u \triangleq c^Tx$, for system (1) such that $[D^TJA_c]_s$ is sign definite or semi-definite depending upon the sign of $[JD]_s$. Thus, we may naturally inquire when a vector $c \in \mathbb{R}^2$ exists such that $D^TJA_c = D^TJA + D^TJbc^T$ has a definite symmetric part. This question is resolved by the following lemma and its corollaries.

__Lemma 2__: For any $Q \in \mathbb{R}^{2x2}$ and $g \in \mathbb{R}^2$, there exists a $c \in \mathbb{R}^2$ such that $[Q + gc^T]_s \geq 0$ if and only if $g_\perp{}^T Q g_\perp \geq 0$, and $[Q + gc^T]_s > 0$ if and only if $g_\perp{}^T Q g_\perp > 0$.

__Proof__:

(i) __Necessity__: If $g_\perp{}^T Q g_\perp < 0$ then $g_\perp{}^T [Q + gc^T]_s g_\perp = g_\perp{}^T Q g_\perp < 0$. If $g_\perp{}^T Q g_\perp = 0$ then $g_\perp{}^T [Q + gc^T]_s g_\perp = 0$.

(ii) __Sufficiency__: If $g_\perp{}^T Q g_\perp = 0$, then $Q = [gd^T]_s$ for some $d \in \mathbb{R}^2$. Hence, if $c \triangleq -d + \gamma g$ for $\gamma \in \mathbb{R}^+$, then $[Q + gc^T]_s = [gd^T + gc^T]_s = \gamma gg^T \geq 0$. Note that any other choice of c leads to an indefinite form for $[Q + gc^T]_s$.

If $g_\perp{}^T Q g_\perp > 0$ then $(g_\perp \pm \varepsilon g)^T Q (g_\perp \pm \varepsilon g) > 0$ for some $\varepsilon > 0$ by continuity. Then choosing $c \triangleq \gamma g$ where $\gamma > \frac{(2+\varepsilon)\|Q\|}{\varepsilon} > 0$ we have $x^T[Q + gc^T]x = \rho^2(\alpha g + g_\perp)^T [Q + gc^T](\alpha g + g_\perp) = \rho^2(\alpha g + g_\perp)^T Q(\alpha g + g_\perp) + \rho^2\alpha^2\gamma g^T g$ which is positive for $|\alpha| < \varepsilon$. Since $|(\alpha g + g_\perp)^T Q(\alpha g + g_\perp)| \leq (\alpha^2 + 2\alpha)\|Q\| g^T g + g_\perp{}^T Q g_\perp$ we have $\rho^2(\alpha g + g_\perp)^T Q(\alpha g + g_\perp) + \rho^2\alpha^2\gamma g^T g \geq \rho^2[g_\perp{}^T Q g_\perp + \gamma\alpha^2 g^T g - (\alpha^2 + 2\alpha)\|Q\| g^T g]$ $\geq \rho^2[g_\perp{}^T Q g_\perp + 2\alpha(\frac{\alpha}{\varepsilon} - 1)\|Q\| g^T g] > 0$ for $|\alpha| \geq \varepsilon$ by choice of γ. □

__Corollary 2.1__: If D is focal then there exists a $c \in \mathbb{R}^2$ such that $[JD]_s$ and $[D^T J A_c]_s$ agree in sign if and only if $|Dx,x| \cdot |AD^{-1}b,b| \geq 0$.

__Proof__: Assume $[JD]_s > 0$. According to Lemma 1, $[D^T J A_c]_s \geq 0$ iff $[D^T Jb]_\perp{}^T D^T JA[D^T Jb]_\perp \geq 0$. But $[D^T Jb]_\perp{}^T D^T JA[D^T Jb]_\perp = b^T J^T DJ^T [D^T JA] JD^T Jb = |D|^2 b^T JAD^{-1} b$ according to Fact B.1. Hence, $[D^T J A_c]_s \geq 0$ iff $0 \leq b^T JAD^{-1} b = |AD^{-1}b,b|$ since $|D|^2 > 0$. The identical proof holds for $[JD]_s < 0$. □

__Corollary 2.2__: If D is focal and $|Dx,x| \cdot |AD^{-1}b,b| > 0$, then, if $\gamma \in \mathbb{R}$, $c \triangleq \gamma D^T Jb$ implies $[JD]_s$ and $[D^T J A_c]_s$ agree in sign when $|\gamma|$ is large enough and sgn γ = = sgn $x^T[JD]_s x$. In this case $D^{-1}A_c$ is focal.

__Proof__: This follows directly from the construction of g in the proof of Lemma 2, when $Q \triangleq D^T JA$. □

Corollary 2.3: If D is focal and $|AD^{-1}b,b| = 0$ $(b \neq 0)$ then $[JD]_s$ and $[D^TJA_c]_s$ agree in sign iff $c \triangleq D^TJ(-d + \gamma b)$ where d is the other eigenvector of AD^{-1} and $\text{sgn}\ \gamma = \text{sgn}\ x^T[JD]_s x$.* In this case $D^{-1}A_c$ is $D^{-1}b$ -critical.

Proof: Again, this follows from the proof of Lemma 2. □

We may now proceed to consider the cases listed above in correspondence with the conditions of Theorem 2.

3. Stabilizability When D is Focal:

According to Lemma 2 and its corollaries, if D is focal, then a c exists such that conditions (ii) and (iii.a) of Theorem 2 hold, when $|AD^{-1}b,b|$ and $[JD]_s$ have the same sign. Surprisingly enough, if $|AD^{-1}b,b| \neq 0$, the same sign condition assures the stabilizability of (A,b) in the sense of LTI pole-placement and, thereby, of (A,b,D) in our sense. The following proposition exploits this coincidence, specifying stabilizability conditions which make implicit use of this fact.**

Proposition 1: If D is focal and $|AD^{-1}b,b| \neq 0$ then there exists a $c \in \mathbb{R}^2$ such that (3) is g.a.s. if and only if $|AD^{-1}b,b| \cdot |Dx,x| > 0$.

Proof: (i) Necessity: According to Corollary 2.1, condition (ii) of Theorem 2 holds only if $|Dx,x| \cdot |AD^{-1}b,b| > 0$ under the assumptions above.

(ii) Sufficiency: Since $|AD^{-1}b,b| \cdot |Dx,x| > 0$ implies condition (ii) and (iii.a) of Theorem 2 according to Corollary 2.2, when $c \triangleq \gamma D^TJb$ where $\text{sgn}\ \gamma = \text{sgn}\ x^TJDx = \text{sgn}\ |Dx,x|$ and $|\gamma|$ is large enough, it remains to show that (i) holds under this choice of feedback.

Since $\text{tr}\{A_c\} = \text{tr}\{A\} + \gamma b^TJ^TDb = \text{tr}\{A\} - \gamma|Db,b|$ and $\gamma|Dx,x| > 0$ for all $x \in \mathbb{R}^2$, we have $\text{tr}\{A\} < 0$ when $|\gamma|$ is large enough.

* If AD^{-1} is b-critical, then there is no "other eigenvector" and the result follows with $d \in (b)$ as seen from the proof of Lemma 2. However, if AD^{-1} is nodal, then we require $d \notin (b)$ in order for this construction to work. Accordingly, in the sequel, the terminology other eigenvector shall designate $d \notin (b)$ if AD^{-1} is nodal, and $d \in (b)$ if AD^{-1} is b-critical.

** The explicit demonstration that $|AD^{-1}b,b| \cdot |Dx,x| > 0$ implies LTI stabilizability is given in Appendix A as a matter of related interest.

Since $|A_c| = |A| + |bc^T| + tr\{J^T cb^T JA\}$ (Fact B.2)

$$= |A| + \gamma b^T JAJ^T D^T Jb$$

$$= |A| + \gamma |D| b^T JAD^{-1} b \text{ (Fact B.1)}$$

$$= |A| + \gamma |D| |AD^{-1}b, b|,$$

we have $|A_c| > 0$ when $|\gamma|$ is large enough, since $|D| > 0$ (D is focal) and $\gamma|AD^{-1}b,b| > 0$ since $\text{sgn } \gamma = \text{sgn } |Dx,x| = \text{sgn } |AD^{-1}b,b|$.

But $tr\{A_c\} < 0$, $|A_c| > 0$ implies A_c stable. □

If D is focal but b is an eigenvector of AD^{-1} then, according to Lemma 2 and its corollaries, $D^{-1}A$ cannot be made focal by any choice of c. By Corollary 2.3, there exists a unique $c \in \mathbb{R}^2$ such that condition (ii) and (iii.a) of Theorem 2 hold, however, there is no guarantee that (A,b) is stabilizable in the sense of LTI pole-placement. Hence, the conditions for stabilizability in this case are more restrictive, and are given as follows.

<u>Proposition 2</u>: Let D be focal, $|AD^{-1}b,b| = 0$ $(b \neq 0)$ and let d be the other eigenvector of AD^{-1} with eigenvalue δ.* Then there exists a $c \in \mathbb{R}^2$ such that (3) is g.a.s. if and only if either

(i) AD^{-1} is b-critical and $|A| > 0$;

or (ii) AD^{-1} is nodal and $\delta|d,b| = \frac{|A|}{|D|}$.

<u>Proof</u>: (i) <u>Necessity</u>: Assume $[JD]_s > 0$ without loss of generality. By Theorem 2, it is necessary that $[D^T JA_c] \geq 0$, and this is true iff $c \triangleq D^T J(-d + \gamma b)$ (where $\text{sgn } \gamma = \text{sgn } |Dx,x|$) according to Corollary 2.3, in which case $D^{-1}A_c$ is $D^{-1}b$-critical. Hence, by condition (iii.a) of Theorem 2, it is necessary that $c_\perp \in (D^{-1}b)$ iff $|A_c| \neq 0$. By construction, $c_\perp = JD^T J(-d + \gamma b) = |D|D^{-1}(d - \gamma b)$ (Fact B.1), hence $c_\perp \in (D^{-1}b)$ iff $d \in (b)$. Thus, we require $|A_c| > 0$ if AD^{-1} is b-critical, and

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* See the footnote to Corollary 2.3.

$0 = |A_c| = |A| + tr\{J^T cb^T JA\}$(Fact B.2) if AD^{-1} is nodal. Since $tr\{J^T cb^T JA\}$ $= |D||AD^{-1}(-d + \gamma b), b| = -|D|\delta|d,b|$, condition (ii), above, follows.

(ii) Sufficiency: By the foregoing construction of c, all requirements of Theorem 2 have been met except the demonstration that $tr\{A_c\} < 0$. But $tr\{A_c\}$ $= tr\{A\} - tr\{bd^T J^T D\} + \gamma\, tr\{bb^T J^T D\}$ and the last term is equal to $-\gamma|Db,b| < 0$ since $\text{sgn}\,\gamma = \text{sgn}\,|Dx,x|$ for all x. Hence $tr\{A_c\} < 0$ when $|\gamma|$ is large enough. □

4. Stabilizability When D is Singular:

If D is nonsingular and not focal, then system (3) violates Theorem 2 as shown by Lemma 1. However, if $|D| = 0$, then for some $d, e \in \mathbb{R}^2$, $D = de^T$. Hence, $c^T xDx = e^T xD'x$, and by choosing $c \in (d_\perp)$, D' in system (2) is d-critical and singular (if $e \in (d_\perp)$ then D is d-critical and singular to begin with). Therefore, when D is singular, it is possible to stabilize (1) in some cases. Before presenting these cases, we state the following useful result.

Lemma 3: If D is singular and d-critical, and $|A| > 0$, then condition (ii) of Theorem 2 holds iff $Ad = \alpha d$ and $\alpha < 0$.

Proof: Since $x^T D^T JAx = |Ax,Dx|$ and $x^T JDx = |Dx,x|$, condition (ii) is equivalent to $0 \leq |Ax,Dx| \cdot |Dx,x| = |A| \cdot |x, A^{-1}dd_\perp^T x| \cdot |dd_\perp^T x,x| = (d_\perp^T x)^2 |A| \cdot |x,A^{-1}d| \cdot |d,x|$ which is true, assuming $|A| > 0$, if and only if $0 \leq |x,A^{-1}d| \cdot |d,x|$. The latter is possible if and only if $A^{-1}d = \alpha d$, $\alpha < 0$. □

In general, when $D = de^T$, $d \notin (e_\perp)$, hence D is nodal as well as singular. In this case stabilizability conditions are quite simple to state.

Proposition 3: If $D = de^T$ is nodal then there exists a $c \in \mathbb{R}^2$ such that (3) is g.a.s. iff $Ad = \alpha d$, $\alpha < 0$.

Proof: (i) Necessity: According to condition (ii) of Theorem 2 we require $c \in (d_\perp)$ or $c \overset{\Delta}{=} \beta Jd$ for D' to not be nodal. In this case condition (iii.b) applies and we require $|A_c| \neq 0$ which necessitates $|A_c| > 0$ according to condition (i) of Theorem 2.

Hence condition (ii) holds iff $A_c d = \alpha d$, $\alpha < 0$ according to Lemma 3. But $A_c d = Ad$ by construction of c, hence $Ad = \alpha d$, $\alpha < 0$.

(ii) Sufficiency: Since $c \triangleq \beta Jd$ satisfies conditions (ii) and (iii) of Theorem 2, it suffices to show that (i) holds for suitable β. Since $\mathrm{tr}\{A_c\} = \mathrm{tr}\{A\} + \beta|d,b|$ and $|A_c| = |A| + \beta b^T JAd = |A| + \alpha\beta|d,b||\beta|$ large and sgn β = −sgn $|d,b|$ implies A_c stable. □

If, however, $d \in (e_\perp)$ then a much greater choice of c is available and stabilizability conditions are more complex.

Proposition 4: If $D = dd_\perp^T$ then there exists a $c \in \mathbb{R}^2$ such that (3) is g.a.s. iff either

(a) b is an eigenvector of A in the null-space of D and (A,b) is a stabilizable pair (in the sense of LTI theory).

or (b) there exists a $\gamma \in \mathbb{R}$ such that

$$\mathrm{tr}\{A\} - \frac{|Ab,d|}{|b,d|} + \gamma < 0$$

$$|A| - \gamma\frac{|Ab,d|}{|b,d|} > 0$$

Proof: According to conditions (i) and (iii.b) of Theorem 2, $|A_c| > 0$ and $A_c^{-1}D = \gamma D$, hence $A_c d = \gamma d$, or $0 = |A_c d,d| = |Ad,d| + c^T d|b,d|$. If $|d,b| = 0$ then we require $0 = |Ad,d| = |Ab,b|$, hence (A,b) is not controllable, and we must have LTI stabilizability, in which case any c such that A_c is stable meets the conditions of Theorem 2. This accounts for case (a), above.

If $|d,b| \neq 0$ then $|A_c d,d| = 0$ iff $c \triangleq (\gamma d_\perp - A^T d_\perp)\frac{1}{|d,b|}$ (for some $\gamma \in \mathbb{R}$). Hence $\mathrm{tr}\{A_c\} = \mathrm{tr}\{A\} - \frac{d_\perp^T Ab}{d_\perp^T b} + \gamma = \mathrm{tr}\{A\} - \frac{|Ab,d|}{|b,d|} + \gamma$, and $|A_c| = |A| - \gamma\frac{d_\perp^T Ab}{d_\perp^T b}$ (Fact B.2) $= |A| - \gamma\frac{|Ab,d|}{|b,d|}$, giving rise to the condition in (b), above. □

5. Summary and Conclusion:

The central result of this paper is the statement of necessary and sufficient conditions for the stabilizability of (1) under constant linear state feedback as given by Theorem 1 in the introduction. As a formal proof of that theorem we may summarize the results of Sections 2-4.

If D is focal and $|AD^{-1}b,b| \neq 0$ then (3) is g.a.s. iff $\text{sgn}\,|AD^{-1}b,b| = \text{sgn}\,|Dx,x|$ according to Proposition 1 (Section 3). In this case a stabilizing controller is given by $c \triangleq \gamma D^T Jb$, $\text{sgn}\,\gamma = \text{sgn}\,|Dx,x|$, and $|\gamma|$ suitably large. If $|AD^{-1}b,b| = 0$ and d is the other eigenvector of AD^{-1} then a stabilizing controller given by $c \triangleq D^T J(-d + \gamma b)$, $\text{sgn}\,\gamma = \text{sgn}\,|Dx,x|$ may be chosen iff the conditions of Proposition 2 (Section 3) hold. Thus, if D is focal, condition (i) of Theorem 1 is necessary and sufficient for stabilizability.

If D is singular and nodal then (3) is g.a.s. iff d, its non-zero eigenvector, is a stable eigenvector of A, according to Proposition 3 (Section 4). In this case $c \triangleq \beta Jd$, $\text{sgn}\,\beta = -\,\text{sgn}\,|d,b|$, $|\beta|$ suitably large is a stabilizing controller. If D is singular and critical then (3) is g.a.s. iff the conditions of Proposition 4 (Section 4) hold. If D is b-critical and those conditions are met, then any c which stabilizes (A,b) in the sense of LTI pole-placement, stabilizes (1). If D is d-critical, and $b \notin (d)$ and the conditions are met then $c \triangleq (\gamma d_\perp - A^T d_\perp)\frac{1}{|d,b|}$ is a stabilizing controller. Thus, if D is singular, condition (ii) of Theorem 1 is necessary and sufficient for stabilizability.

If D is neither focal, nor singular then (3) is never g.a.s. according to Lemma 1 (Section 2). Thus Theorem 1 lists complete necessary and sufficient conditions for stabilizability, as claimed.

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Appendix A

Stabilizability of (A,b,D) and (A,b)

In Section 3 it is shown that $|Dx,x| \cdot |AD^{-1}b,b| > 0$ implies (A,b,D) may be stabilized by $c \triangleq \gamma D^T Jb$. This result, given as Proposition 1, implicitly makes use of the following.

Fact: If $|Dx,x| \cdot |AD^{-1}b,b| > 0$ then (A,b) is a stabilizable pair.

Proof: Without loss of generality assume $|Dx,x| > 0$, and apply an identical proof for the case $|Dx,x| < 0$. If $|Ab,b| \neq 0$ then (A,b) is controllable, and hence stabilizable. If $|Ab,b| = 0$ then $Ab = \beta b$ $\beta \in R$ and $Ad = \delta d$ for some $d \in \mathbb{R}^2$, $\delta \in \mathbb{R}$.

If $d \notin (b)$ then $D^{-1}b = \gamma_1 d + \gamma_2 b$ where $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 \neq 0$ since $D^{-1}b \notin (b)$. Hence $|D^{-1}b,b| = |D^{-1}||b,Db| = -|D^{-1}||Db,b| < 0$ since $|D^{-1}| > 0$. Moreover $|D^{-1}b,b| = \gamma_1|d,b| + \gamma_2|b,b| = \gamma_1|d,b|$. But $|AD^{-1}b,b| = \gamma_1|Ad,b| + \gamma_2|Ab,b|$ $= \gamma_1\delta|d,b| = \frac{-\delta}{|D^{-1}|}|Db,b| > 0$ iff $\delta < 0$ in which case (A,b) is stabilizable.

If $d \in (b)$ then $\delta = \beta$ and if $e \in \mathbb{R}^2$ is the generalized eigenvector of A - i.e. $Ae = \beta e + b$ - then we may write $D^{-1}b = \gamma_1 e + \gamma_2 b$ to get the identical result as above. $\square$

Appendix B

Some Useful Properties of 2x2 Matrices

The following two properties are used extensively in Sections 2-4, and follow from direct computation. Define $J \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then

Fact B.1. $|D| \neq 0$ implies $D^{-1} = \frac{1}{|D|} J^T D^T J = \frac{1}{|D|} J D^T J^T$

Fact B.2. $|A + D| = |A| + |D| + \mathrm{tr}\{J^T D^T J A\}$

$$= |A| + |D| + \mathrm{tr}\{J^T A^T J D\}$$

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